Bankruptcy Problem under Uncertainty of Claims and Estate

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Motivation - Introduction

• Several individuals hold claims on a finite resource - estate and the total amount is not enough to fulfill all of the claims

• **Problem:** Distribute the resource (Estate) to individual claimants fairly so as to respect individual claims as much as possible
Classical bankruptcy problems (CB) and games

CBP - triple \((N, c, E)\):
\(N = \{1,2,\ldots,n\}\) – set of claimants
\(c = (c_1,c_2,\ldots,c_n)\) – positive vector of claims \(c_i\), \(i \in N\)
\(E\) - positive total estate.

Alternatively:
\((c; E)\) generates a cooperative game \((N; v)\), - bankruptcy game, whose characteristic form is given by
\[ v(T) = \max\{0, E - \sum c_i\}, \quad T \subseteq N \] - value of coalition \(T\)
Interval bankruptcy problem 1

- \( I(\mathbb{R}) \) - set of all closed and bounded intervals on \( \mathbb{R} \)
- \( \mathbb{R} \) - set of real numbers
- \( I(\mathbb{R})^n \) - set of all \( n \)-dimensional vectors in \( I(\mathbb{R}) \)
- \( I, J \in I(\mathbb{R}) \), with \( I = [I^-; I^+] \), \( J = [J^-; J^+] \) and \( k \geq 0 \)
- Interval operations:
  \[
  I + J = [I^- + J^-; I^+ + J^+], \quad kI = [kI^-; kI^+]
  \]
- Partial ordering on \( I(\mathbb{R})^n \):
  \[
  I \leq J \quad \text{if} \quad I^- \leq J^- \quad \text{and} \quad I^+ \leq J^+
  \]
  \[
  I = J, \quad \text{if} \quad I \leq J \quad \text{and} \quad J \leq I, \quad \text{if} \quad I \leq J \quad \text{and} \quad I \neq J
  \]
For any \( T \subseteq \mathbb{N} \), we use the notation:
\[
\hat{c}^-(T) = \sum_{i \in T} c_i^-, \quad \hat{c}^+(T) = \sum_{i \in T} c_i^+
\]
Minimal/maximal rights:
\[
m_i^- (e) = \max\{c_i^-, e - \hat{c}^+(N \setminus \{i\})\}, \quad m_i^+ (e) = \min\{c_i^+, e - c(N \setminus \{i\})\}
\]
**Definition 1:** A *bankruptcy rule* for an IB-problem \((c, E)\) is a vector mapping \(s : \mathcal{I}(\mathbb{R}^+)^{n+1} \to \mathcal{I}(\mathbb{R}^+)^n\) where \(s(c, E) = (s_1(c, E), \ldots, s_n(c, E))\), such that \(c = (c_1, \ldots, c_n) \in \mathcal{I}(\mathbb{R}^+)^n\), \(c_i = [c_i^-; c_i^+]\), \(i \in N\), and \(E = [E^-; E^+] \in \mathcal{I}(\mathbb{R}^+)\), satisfying

\[(1) \quad s_i(c, E) = [s_i^-(c, E), s_i^+(c, E)] \subseteq c_i = [c_i^-; c_i^+], \quad \text{for all } i \in N, \quad (\text{Individual rationality})\]

\[(2) \quad E = [E^-; E^+] \subseteq \sum_{j \in N} s_j(c, E). \quad (\text{Efficiency})\]
Proposition 1: \((c, E)\) - IB-problem. Let 
\[c(N) \leq E^- \leq E^+ \leq c^+(N).\]
Then \(s_i(c, E) = [s_i^-, s_i^+] \in I(\mathbb{R}^+)\) defined for \(i \in N\), by

\[
s_i^- = m_i^- (E^-) + [m_i^+ (E^-) - m_i^- (E^-)] \frac{E^- - m_N^- (E^-)}{m_N^+ (E^-) - m_N^- (E^-)}, \quad (*)
\]

\[
s_i^+ = m_i^- (E^+) + [m_i^+ (E^+) - m_i^- (E^+)] \frac{E^+ - m_N^- (E^+)}{m_N^+ (E^+) - m_N^- (E^+)}, \quad (**) 
\]
is a bankruptcy rule called the \emph{adjusted proportional rule (AP-rule)} for the IB-problem \((c, E)\) satisfying conditions (1), (2).
Fuzzy interval bankruptcy problem 1

- Claimants declare their claims with vague words: “about”, “around”, “rather small”, “very big”, etc.
- **The key issue**: how to distribute the uncertain, i.e. interval, fuzzy interval or, eventually, the estate given with some probability, to the individual claimants?
- A *fuzzy set* $A$ of $\mathbb{R}$ is a *fuzzy number* (fuzzy interval), whenever $A$ is normal (i.e. there exists $x_0$ with $\mu_A(x_0) = 1$) and its membership function $\mu_A : \mathbb{R} \to [0;1]$ satisfies that the $\alpha$-cut $[A]_\alpha = \{x \mid \mu_A(x) \geq \alpha\}$ is closed, compact and convex subset of $\mathbb{R}$ for every $\alpha \in [0;1]$.
- Fuzzy number $A$ of $\mathbb{R}$ is equivalent to the family of $\alpha$-cuts $\{[A]_\alpha \mid \alpha \in [0;1]\}$.
Fuzzy interval bankruptcy problem 2

**Definition 2:** \( \tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_n) \in F(\mathbb{R}^+)^n \) be a vector of fuzzy numbers:

\( \tilde{c}_i = [c_i^-(\alpha); c_i^+(\alpha)], \ i \in \mathbb{N}, \ \tilde{E} = [E^-(\alpha); E^+(\alpha)] \in F(\mathbb{R}^+) \)

\( \alpha \in [0;1] \) be the families of \( \alpha \)-cuts

A **bankruptcy rule** for an FB-problem \((\tilde{c}, \tilde{E})\) is a vector mapping \( \tilde{s} : F(\mathbb{R}^+)^{n+1} \to F(\mathbb{R}^+)^n : \)

\[ [\tilde{s}(\tilde{c}, \tilde{E})]_\alpha = ([\tilde{s}_1(\tilde{c}, \tilde{E})]_\alpha, \ldots, [\tilde{s}_n(\tilde{c}, \tilde{E})]_\alpha) \]

where \( \tilde{s}_i : F(\mathbb{R}^+)^{n+1} \to F(\mathbb{R}^+), \ i \in \mathbb{N} \).

Here, for each \( \alpha \in [0;1], \ [\tilde{s}(\tilde{c}, \tilde{E})]_\alpha \) is an IB-problem.
Proposition 2: Let $(\tilde{c}; \tilde{E})$ be a FB-problem. Let

\[ \tilde{E} = \{ [E^-(\alpha); E^+(\alpha)] | \alpha \in [0;1] \} \]

and let

\[ \sum_{i \in S} c_i^-(\alpha) \leq E^-(\alpha) \leq E^+(\alpha) \leq \sum_{i \in S} c_i^+(\alpha) \]

for all \( \alpha \in [0;1] \).

Then for \( \alpha \in [0;1] \), \( \tilde{s}_i(\alpha) = [s_i^-(\alpha); s_i^+(\alpha)] \in l(\mathbb{R}^+) \)

is a closed interval defined for \( i \in N \), by

\[ s_i^-(\alpha) = m_i^-(E^-(\alpha)) + [m_i^+(E^-(\alpha)) - m_i^-(E^-(\alpha))] \frac{E^-(\alpha) - m_N(E^-(\alpha))}{m_N(E^-(\alpha)) - m_N(E^-(\alpha))}, \quad (+) \]

\[ s_i^+(\alpha) = m_i^-(E^+(\alpha)) + [m_i^+(E^+(\alpha)) - m_i^-(E^+(\alpha))] \frac{E^+(\alpha) - m_N(E^+(\alpha))}{m_N(E^+(\alpha)) - m_N(E^+(\alpha))}. \quad (+++) \]

Family \{ [s_i^-(\alpha); s_i^+(\alpha)] | \alpha \in [0;1] \}, where the \( \alpha \)-cuts are defined by (+), (+++), defines a bankruptcy rule called the \textit{adjusted fuzzy proportional rule} (AFP-rule) for the FB-problem \((\tilde{c}; \tilde{E})\).
Fuzzy interval bankruptcy problem 4

The mean values $s_i^- (\widehat{E}), s_i^+ (\widehat{E})$ give the corresponding interval share $[s_i^- (\widehat{E}); s_i^+ (\widehat{E})]$ of claimant $i$:

\[
\begin{align*}
    s_i^- (\widehat{E}) &= \frac{\int_0^1 \alpha s_i^- (\alpha) d\alpha}{\int_0^1 s_i^- (\alpha) d\alpha}, \\
    s_i^+ (\widehat{E}) &= \frac{\int_0^1 \alpha s_i^+ (\alpha) d\alpha}{\int_0^1 s_i^+ (\alpha) d\alpha},
\end{align*}
\]

$i \in N. \quad (2)$

Moreover, $S_i (\widehat{E})$ is the crisp corresponding division share of claimant $i \in N$, defined by

\[
S_i (\widehat{E}) = \frac{s_i^- (\widehat{E}) + s_i^+ (\widehat{E})}{2}, \quad i \in N. \quad (3)
\]
Example: 1

Let $({\tilde{c}}; {\tilde{E}})$ be a FB-problem as follows. The following claims are expressed as trapezoidal fuzzy intervals (fuzzy numbers):

- $N = \{1, 2, 3\}$, $\tilde{c} = ({\tilde{c}}_1, {\tilde{c}}_2, {\tilde{c}}_3) \in F(R^+)^3$, where
  - $\tilde{c}_1 = [c_{11}; c_{12}; c_{13}; c_{14}] = [10; 25; 25; 35],$
  - $\tilde{c}_2 = [c_{21}; c_{22}; c_{23}; c_{24}] = [25; 35; 35; 50],$
  - $\tilde{c}_3 = [c_{31}; c_{32}; c_{33}; c_{34}] = [30; 40; 40; 60].$

- Fuzzy estate is also a trapezoidal fuzzy number
  - $\tilde{E} = [E_1; E_2; E_3; E_4] = [85; 100; 100; 115].$

- For $\alpha \in [0; 1]$ the equivalent formulas by $\alpha$-cuts are as follows
  - $\tilde{c}_1 = [10 + 15\alpha; 35 - 10\alpha]$, $\tilde{c}_2 = [25 + 10\alpha; 50 - 15\alpha],$
  - $\tilde{c}_3 = [30 + 10\alpha; 60 - 20\alpha].$
  - $\tilde{E} = [85 + 15\alpha; 115 - 15\alpha]$, see Fig. 1.
Example: 2 (Fig. 1)

Fuzzy number claims

![Graph showing fuzzy number claims with lines for s1-, s1+, s2-, s2+, s3-, and s3+.](image-url)
Example: 3 (Fig. 2)

Fuzzy number shares

\[
S_1 = 22.32 \\
S_2 = 31.65 \\
S_3 = 44.02
\]
Example: 4

- Substituting these values into formulas (2) and (3), we obtain functions $s_i^- (\alpha)$ and $s_i^+ (\alpha)$, see Fig. 2. Hence, by these formulas we calculate the integrals of the interval share of each claimant $i$ as

- $[s_1^- (\tilde{E}); s_1^+ (\tilde{E})] = [19,00 ; 25,25]$, 
- $[s_2^- (\tilde{E}); s_2^+ (\tilde{E})] = [31,00 ; 37,25]$, 
- $[s_3^- (\tilde{E}); s_3^+ (\tilde{E})] = [36,00 ; 43,50]$. 

- Moreover, by (3) we obtain the crisp division share of each claimant $i \in N = \{1,2,3\}$, as

- $S_1 (\tilde{E}) = 22,32 ; S_2 (\tilde{E}) = 31,65 ; S_3 (\tilde{E}) = 44,02$. 
- The above mentioned division scheme $s(, \tilde{E})$ is an interval solution of the given FB-problem $(\tilde{c}, \tilde{E})$. Moreover, by the vector of mean values $S(\tilde{E}) = (22,32, 31,65, 44,02)$ we obtain a crisp solution of FB-problem $(\tilde{c}, \tilde{E})$. 

Conclusion

• When claims of claimants had fuzzy interval uncertainty, we settled such type of division problems by transforming it into division problems under classical interval uncertainty.

• An example was presented to illustrate particular problems and solution concepts. Here, we extended the classical bankruptcy problem (CB-problem), and the corresponding proportional rule (AP-rule) to FB-problem.

• The other classical bankruptcy rules, e.g. contested garment consistent rule (CGC-rule) and recursive completion rule (RC-rule) could be also extended to FB-problem in the future research.
Some references